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THE ORIENTED BORDISM OF Z_{2^k} ACTIONS

BY

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ABSTRACT. Let R_2 be the subring of the rationals given by $R_2 = Z[\frac{1}{2}]$. It is shown that for $G = Z_{2^k}$ the bordism group of orientation preserving G actions on oriented manifolds tensored with R_2 is a free $\Omega_* \otimes R_2$ module on even dimensional generators (where Ω_* is the oriented bordism ring).

1. **Introduction.** Let G be a group. Let Ω_*^G denote the bordism group of differentiable orientation preserving G actions on closed oriented manifolds. In R. E. Stong's paper [10] Ω_*^G is understood for G a p-group and p an odd prime. In [9] H. L. Rosenzweig shows that $\Omega_*^{Z_2} \otimes Q = 0$ if $* \neq 4k$. In this paper the module structure of Ω_*^G is determined up to 2-torsion for $G = Z_{2^k}$.

§§ 2, 3, and 4 are largely preliminary material. In §5 it is shown that $\Omega_*^{Z_{2^k}} \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even dimensional generators (where $R_n = \{a/b \mid a$ is an integer and b is a power of $n\}$ is a subring of the rationals).

This paper discusses part of the research undertaken while I was a Ph. D. candidate at the University of Virginia. I would like to express my appreciation to my advisor, R. E. Stong, who directed this research in a most generous way.

2. **Equivariant bordism.** For a finite abelian group G a family F'' of subgroups of G is a collection of subgroups of G such that if $H \in F''$ and $K < H$, then $K \in F''$. If (M, σ) is a manifold with G action, then (M, σ) is F''-free if for each $x \in M$, the isotropy subgroup of x is an element of F''.

Let $F' \subset F''$ be families of subgroups of G. Let (X, A) be a space pair with G action. Consider 5-tuples (M, M_0, M_1, σ, f) where

(1) M, M_0, M_1 are compact differentiable oriented manifolds with n the dimension of M.

(2) $\partial M = M_0 \cup M_1$, $\partial M_0 = \partial M_1 = M_0 \cap M_1$.

(3) $\sigma: G \times M \to M$ is a differentiable G-action which preserves M_0 and M_1 and which preserves the orientation on M.

(4) (M, σ) is F''-free while $(M_0, \sigma/G \times M_0)$ is F'-free.

(5) $f: (M, M_1) \to (X, A)$ is an equivariant map.

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Under the usual equivariant bordism relation (see [10, §2]) one forms a set of equivalence classes of such 5-tuples, denoted $\Omega_n^G(\mathcal{F}'', \mathcal{F}')(X, A)$, with an abelian group structure induced by disjoint union. The graded sum of these groups has an Ω_* module structure induced by cartesian product and is denoted by $\Omega_*^G(\mathcal{F}'', \mathcal{F}')(X, A)$.

Now if $h: (X, A) \to (Y, B)$ is an equivariant map between spaces with G action, one has an induced homomorphism $h_*: \Omega_*^G(\mathcal{F}'', \mathcal{F}')(X, A) \to \Omega_*^G(\mathcal{F}'', \mathcal{F}')(Y, B)$ sending $[M, M_0, M_1, \sigma, f]$ into $[M, M_0, M_1, \sigma, h \circ f]$. Let $\varnothing$ denote the empty set. Then there is a degree -1 boundary map $\partial_*: \Omega_*^G(\mathcal{F}'', \mathcal{F}')(X, A) \to \Omega_*^G(\mathcal{F}'', \mathcal{F}')(A, \varnothing) \equiv \Omega_*^G(\mathcal{F}'', \mathcal{F}')(A)$ sending $[M, M_0, M_1, \sigma, f]$ into $[M_1, \partial M_1, \varnothing, \sigma/G \times M_1, f/M_1]$. From [10, Proposition 2.1], $\Omega_*^G(\mathcal{F}'', \mathcal{F}')(-)$ and ∂_* define an equivariant homology theory on the category of G-pairs to the category of Ω_* modules. Specifically this theory satisfies equivariant homotopy, excision, and exactness axioms.

From [10, Proposition 2.2], one knows that for families of subgroups $\mathcal{F}' \subset \mathcal{F}''$ there is an exact triangle

$$\begin{array}{ccc} \Omega_*^G(\mathcal{F}')(X, A) & \xrightarrow{\alpha_*} & \Omega_*^G(\mathcal{F}'')(X, A) \\ {\scriptstyle \partial'_*}\nwarrow & & \swarrow{\scriptstyle \beta_*} \\ & \Omega_*^G(\mathcal{F}'', \mathcal{F}')(X, A) & \end{array}$$

in which α_* and β_*, respectively, forget $\mathcal{F}'$ and $\mathcal{F}''$-freeness while ∂'_* sends $[M, M_0, M_1, \sigma, f]$ into $[M_0, \varnothing, \partial M_0, \sigma/G \times M_0, f/M_0]$.

Note. If G is an abelian group and $H < G$, the collection of all subgroups of H is a family of subgroups of G. If $H \lneqq G$, this family is denoted by $\mathcal{F}_H$. In particular $\mathcal{F}_e$ denotes the family consisting of the identity subgroup. Let $\mathcal{F}$ denote the family of all subgroups of G.

3. **Classifying spaces for bundles with G action.** Let G be a finite abelian group with exactly r distinct irreducible complex representations. Let $C^\infty = C_1^\infty \oplus C_2^\infty \oplus \cdots \oplus C_r^\infty$. Define a G action on C^∞ by considering C_i^∞ as a countable direct sum of the ith irreducible representation. Now let BU_s be the Grassmannian of complex s-planes in C^∞ and γ_s be the universal complex s-plane bundle over BU_s. Since the elements of G act on C^∞ via complex linear transformations, there is an induced G action on BU_s and γ_s (see [10, §3]). One learns from Atiyah [2, §1.6] that $\gamma_s \to BU_s$ is the universal complex n-plane bundle in the category of G-spaces.

One can perform essentially the same construction in the real case by taking the Grassmannian of real n-planes in C^∞. In this way one gets BO_s together with its canonical bundle γ_s, the universal real s-plane bundle in the category of G-spaces. Note that in what follows these G-spaces are called BU_s and BO_s

except in cases where the context does not make the meaning clear. In these cases the notation (BU_s, G) and (BO_s, G) is used.

In the process of defining BU_s and BO_s together with their canonical bundles one may place a metric on the γ_s such that the G action is orthogonal with respect to this metric. Further, for any G-bundle $E \to X$ of dimension s over a compact Hausdorff space X, one may assume there is a metric on E such that

(a) the G action on E is orthogonal with respect to this metric,

(b) the bundle map covering the classifying map takes

$$(D(E), S(E)) \to (D(\gamma_s), S(\gamma_s))$$

where $D(-)$ denotes the unit disc bundle and $S(-)$ denotes the unit sphere bundle.

Now consider the G-spaces BO_s and BU_s and the fixed sets of subgroups of G acting on BO_s and BU_s. Let $H < G$ and X be a compact Hausdorff G-space. The isomorphism classes of G-bundles over X of real dimension s, $\text{vect}_s^G(X)$, are in 1-1 correspondence with the G-homotopy classes of equivariant maps from X into BO_s, $[X, BO_s]_G$. Now if $H < G$ fixes X, any equivariant map $X \to BO_s$ goes into the fixed set of H acting on BO_s, $F_H(BO_s)$. Hence if H fixes X, $\text{vect}_s^G(X) \leftrightarrow [X, F_H(BO_s)]_G$. It follows that $F_H(BO_s)$ is the classifying space of G bundles of dimension s over base spaces X such that H fixes X. Exactly the same analysis is true for complex s-bundles over X and $F_H(BU_s)$.

Further, if $E \to X$ is a complex G bundle and $H < G$ fixes X, E splits into G subbundles according to the nontrivial irreducible complex representations of H [2, §1.6]. The classifying space for G-bundles over a base which H fixes can be understood in terms of these subbundles. Using this information one can compute explicitly the fixed sets $F_H(BU_s)$. Using similar techniques one can understand $F_H(BO_s)$. In particular, for the purposes of this paper one records the following computations.

PROPOSITION 3.1. *If $H < G$ with $d =$ the order of H, then $F_H(BU_s, G)$ is G homotopy equivalent to $\bigcup BU_{t_1} \times \cdots \times BU_{t_d}$ where $\Sigma\, t_i = s$.* □

Since the real irreducible representations of Z_2 are multiplication by $+1$ and by -1 on one-dimensional vector spaces, a Z_{2k} bundle E over a Z_{2k} space which is fixed by Z_2 decomposes into $E_1 \oplus E_{-1}$ where Z_2 acts in the fibers of E_i by multiplication by i. Thus the classifying space for s-dimensional real vector bundles over Z_{2k} spaces fixed by Z_2 is $\bigcup BO_{t_1} \times BO_{t_{-1}}$ where $t_{-1} + t_1 = s$. Thus

PROPOSITION 3.2. $F_{Z_2}(BO_s, Z_{2^k})$ *is* Z_{2^k} *homotopy equivalent to* $\bigcup BO_{t_{-1}} \times BO_{t_1}$. □

It is evident that the component of $F_{Z_2}(BO_s, Z_{2^k})$ above which Z_2 acts as -1 in the fibers of the canonical bundle is a BO_s. Denote this component by $F^-_{Z_2}(BO_s, Z_{2^k})$. The Z_{2^k} action restricted to $F_{Z_2}(BO_s, Z_{2^k})$ can be considered a $Z_{2^{k-1}}$ action. If $k > 1$ it is necessary to know the fixed set of $Z_{2^j} < Z_{2^{k-1}}$ acting on $F^-_{Z_2}(BO_s, Z_{2^k})$. $F_{Z_{2^j}}[F^-_{Z_2}(BO_s, Z_{2^k})]$ is the classifying space for Z_{2^k} bundles $E \to X$ which have the properties

(a) $Z_{2^{j+1}} < Z_{2^k}$ fixes X.

(b) $Z_2 < Z_{2^{j+1}} < Z_{2^k}$ acts on the fibers of E as multiplication by -1.

For such a bundle E splits into subbundles with respect to the irreducible representations of $Z_{2^{j+1}}$. Since each irreducible representation of $Z_{2^{j+1}}$ which satisfies (b) is the realification of an irreducible complex representation, each of the subbundles of E has a complex structure. Thus if there are r irreducible real representations of $Z_{2^{j+1}}$ satisfying (b) one has

PROPOSITION 3.3. $F_{Z_{2^j}}[F^-_{Z_2}(BO_s, Z_{2^k})]$ *is* $Z_{2^{k-1}}$ *homotopy equivalent to* $\bigcup BU_{t_1} \times \cdots \times BU_{t_r}$ *with* $\Sigma\, t_i = s$. □

4. **A special case of equivariant transverse regularity.** Let γ_{2s} represent the canonical $2s$ plane bundle over $F^-_{Z_2}(BO_{2s}, Z_{2^k})$. (Note that $(BO_{2s}, Z_{2^{k-1}})$ is $Z_{2^{k-1}}$ homotopy equivalent to $F^-_{Z_2}(BO_{2s}, Z_{2^k})$.) Since $Z_2 < Z_{2^k}$ acts as multiplication by -1 in the fibers of γ_{2s} and since the determinant of -1 acting on an even dimensional vector space is $+1$, the Z_2 action dies when one takes the determinant bundle of γ_{2s} together with its induced action. In other words, $\det \gamma_{2s} \to F^-_{Z_2}(BO_{2s}, Z_{2^k})$ is a $Z_{2^{k-1}}$ bundle.

PROPOSITION 4.1. *If*

$$f\colon (M, \partial M, Z_{2^{k-1}}\ \textit{action}) \to (D(\det \gamma_{2s}), S(\det \gamma_{2s}), \det(Z_{2^k}\ \textit{action}))$$

is an equivariant map, then f *may be equivariantly homotoped to be transverse regular on the zero section of* $\det \gamma_{2s}$. *Further, if* $A \subset M$ *is a closed subspace and if* f/A *is already transverse regular, the homotopy can be chosen to fix* A.

PROOF. One needs only to check that the hypotheses for Lemma 4.2 in [10] are satisfied. Therefore one looks at the fixed set of $Z_{2^j} < Z_{2^{k-1}}$ acting on $F^-_{Z_2}(BO_{2s}, Z_{2^k})$ for all $1 \leq j \leq k-1$, and one checks that if $x \in BO_{2s}$ is fixed by Z_{2^j}, then v is fixed by Z_{2^j} for all $v \in \det \gamma_{2s}/x$.

If T is the generator of $Z_{2^{j+1}}$ acting on γ_{2s}, T acts as a real linear transformation on γ_{2s}/x such that T^{2^j} acts as multiplication by -1. Further, the minimum polynomial of T, m_T, must divide $y^{2^{j+1}} - 1 = (y-1)\cdot(y+1) \cdot q_1(y) \cdot \cdots \cdot q_{2^j-1}(y)$ where $q_i(y)$ is an irreducible quadratic of the form

$y^2 + ay + 1$. $y - 1$ does not divide m_T since this would imply that T is multiplication by 1 on some one-dimensional subspace of γ_{2s}/x. Elementary linear algebra then yields that $\det T = +1$ which implies that $\det T$ fixes pointwise the fiber $\det \gamma_{2s}/x$. □

5. **The oriented bordism of Z_{2^k}.** For a group G, denote by Ω_*^G the equivariant bordism module $\Omega_*^G(\mathfrak{F})(pt)$. In this section $\Omega_*^{Z_{2^k}} \otimes R_2$ is computed. Let (X, A) be a c.w. pair with Z_{2^k} action having the property that $F_{Z_{2^j}}(X, A)$ is a c.w. pair for $0 \leqslant j \leqslant k$ where $F_{Z_{2^j}}(X, A)$ is the fixed set of Z_{2^j} acting on (X, A). For a bundle E with unit disc, $D(E)$, and unit sphere, $S(E)$, one denotes by $T(E)$ the space $D(E)/S(E)$, the Thom space of E. The primary tool of this paper is the following theorem.

THEOREM 5.1. *$\Omega_*^{Z_{2^k}}(\mathfrak{F}, \mathfrak{F}_e)(X, A)$ is isomorphic to*

$$\bigoplus_{s=0}^{[*/2]} \tilde{\Omega}_{*-2s+1}^{Z_{2^{k-1}}}(\mathfrak{F})(F_{Z_2}(X)/F_{Z_2}(A) \wedge T(\det \gamma_{2s}))$$

where γ_{2s} is the canonical $2s$ plane bundle over $F_{Z_2}^-(BO_{2s}, Z_{2^k})$.

PROOF. Let $[M, M_0, M_1, T, f] \in \Omega_*^{Z_{2^k}}(\mathfrak{F}, \mathfrak{F}_e)(X, A)$ where T generates the Z_{2^k} action on M. Let F_2 be the $(n - s)$-dimensional component of the fixed set of $Z_2 < Z_{2^k}$ acting on M. Then F_2 is a submanifold of M with an induced action of $Z_{2^{k-1}}$ which is covered in the normal bundle to F_2 in M, ν, by an action of Z_{2^k}. Further, $\partial F_2 = F_2 \cap M_1$. Since one may identify the disc of the normal bundle equivariantly with a small tubular neighborhood of F_2, one knows that no elements of the disc of the normal bundle $-$ {zero section} can be fixed by Z_{2^j} for $1 \leqslant j \leqslant k$. Since each fiber of ν is a representation space for Z_2, ν is a Z_{2^k} bundle over F_2 such that Z_2 acts as -1 in the fibers. One then knows that $\nu \to F_2$ is classified equivariantly into $F_{Z_2}^-(BO_{2s}, Z_{2^k})$ yielding a Z_{2^k} bundle map

$$\begin{array}{ccc} \nu & \xrightarrow{g'} & \gamma_{2s} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ F_2 & \xrightarrow{g} & BO_{2s} \end{array}$$

By taking the determinant bundles of ν and γ_{2s} one gets a similar diagram of $Z_{2^{k-1}}$ bundle maps.

One may assume that $\det g'$ maps the (D, S) pair of $\det \nu$ into the (D, S) pair of $\det \gamma_{2s}$. Letting $\tilde{\pi}$: $\det \nu \to F_2$ be the projection, and crossing $\det g'$ with $f \circ \tilde{\pi}$, one gets a map from the pair

$$(D(\det \nu), D(\det \nu/\partial F_2) \cup S(\det \nu))$$

into

$$(F_{Z_2}(X, A) \times (D(\det \gamma_{2s}), S(\det \gamma_{2s}))).$$

Since the first Stiefel-Whitney classes of ν and the tangent bundle of F_2, $\tau(F_2)$, are equal, $D(\det \nu)$ is an oriented manifold. Let T' generate the $Z_{2^{k-1}}$ action on $D(\det \nu)$. One notes that T' is orientation preserving if $\det(dT') = T' \times 1$ on $\det \tau(D(\det \nu))$ [6, Lemma 3]. Since $\det dT' = \widetilde{\pi}^*(\det dT)$ on $\widetilde{\pi}^*(\det \tau(M)/F_2) \cong \det \tau(D(\det \nu))$ it follows that T' is orientation preserving since T is orientation preserving. Thus by summing over the discs of the determinant bundles of all possible components of the fixed set of Z_2, one may define a map

$$F: \Omega_*^{Z_{2^k}}(\mathcal{F}, \mathcal{F}_e)(X, A)$$

$$\to \bigoplus_{s=0}^{[*/2]} \Omega_{*-2s+1}^{Z_{2^{k-1}}}(\mathcal{F})(F_{Z_2}(X, A) \times (D(\det \gamma_{2s}), S(\det \gamma_{2s}))).$$

In order to define an inverse to F consider

$$[N, \partial N, S, h] \in \Omega_*^{Z_{2^{k-1}}}(\mathcal{F})(F_{Z_2}(X, A) \times (D(\det \gamma_{2s}), S(\det \gamma_{2s}))).$$

One has $N \xrightarrow{p_2 \circ h} D(\det \gamma_{2s})$ and $p_2 \circ h$ is an equivariant map which, by Proposition 4.1, may be considered to be transverse regular on the zero section, BO_{2s}, of $\det(\gamma_{2s})$. Let $N' = (p_2 \circ h)^{-1}(BO_{2s})$. Since γ_{2s} has a Z_{2^k} action covering the $Z_{2^{k-1}}$ action on BO_{2s}, $(p_2 \circ h)^*(\gamma_{2s}) \xrightarrow{\pi'} N'$ is a bundle with an induced Z_{2^k} action such that $Z_2 < Z_{2^k}$ acts as -1 in the fibers. Let S' generate the Z_{2^k} action on $(p_2 \circ h)^*(\gamma_{2s})$. $D((p_2 \circ h)^*(\gamma_{2s}))$ is oriented and one checks that $\det dS'$ acts as $S' \times 1$ on the determinant of the tangent bundle. Hence S' is orientation preserving by [6, Lemma 3]. Hence by mapping $[N, \partial N, S, h]$ into

$$[D((p_2 \circ h)^*(\gamma_{2s})), S(p_2 \circ h)^*(\gamma_{2s}), D((p_2 \circ h)^*(\gamma_{2s})/\partial N'), S', p_1 \circ h \circ \pi']$$

one defines a map K from

$$\Omega_*^{Z_{2^{k-1}}}(\mathcal{F})(F_{Z_2}(X, A) \times (D(\det \gamma_{2s}), S(\det \gamma_{2s})))$$

into $\Omega_{*+2s-1}^{Z_{2^k}}(\mathcal{F}, \mathcal{F}_e)(X, A)$.

To see that $F \circ K = \mathrm{id}$ one notes that $D[\det(p_2 \circ h)^*(\gamma_{2s})]$ may be regarded as a tubular neighborhood of N' in N. By a deformation one may assume that $p_2 \circ h$ maps

$$\{N - [D(\det(p_2 \circ h)^*(\gamma_{2s})) - S(\det(p_2 \circ h)^*(\gamma_{2s}))]\}$$

into $S(\det \gamma_{2s})$. Let π'' be the bundle projection, $\pi''\colon (p_2 \circ h)^*(\det \gamma_{2s}) \to N'$. Since N' is a strong equivariant homotopy retract of its tubular neighborhood, there is an equivariant homotopy $J\colon N \times I \to F_{Z_2}(X)$ giving a homotopy between $p_1 \circ h$ and J_1 where J_1 has the property that J_1 on $D((p_2 \circ h)^*(\det \gamma_{2s}))$ is given by $(p_1 \circ h/N') \circ \pi''$. It follows that

$$\{N \times I, \partial N \times I \cup N \times 1 - \operatorname{int} D((p_2 \circ h)^*(\det \gamma_{2s})), S \times 1, (J \times (p_2 \circ h)) \times 1\}$$

gives a bordism between $[N, \partial N, S, h]$ and $F \circ K([N, \partial N, S, h])$.

To see that $K \circ F = \mathrm{id}$ it suffices to observe that F_2 is a strong equivariant retract of its tubular neighborhood, $D(\nu)$, and hence one may suppose f is homotopic to a map H such that $H/D(\nu) = f/F_2 \circ \pi$. Now $K \circ F$ is obtained by restricting to $D(\nu)$. Since Z_{2^k} acts freely in the complement of F_2, $[M, M_0, M_1, T, f] = K \circ F([M, M_0, M_1, T, f])$. □

Now suppose (X, A) is a c.w. pair acted on by $G = Z_{2^k}$. Let q denote the quotient map onto the space pair $(X/G, A/G)$ obtained by identifying the orbits of the G action. It is a well-known fact that $q^*: H^*(X/G, A/G; R_2) \to H^*(X, A; R_2)$ is a monomorphism onto the elements of $H^*(X, A; R_2)$ which are invariant under the G action (see [5, Corollary 2.3]). This fact together with the appropriate universal coefficient theorem indicates that if $H_*(X, A; R_2)$ is a free R_2 module on even [odd] dimensional generators, then $H_*(X/G, A/G; R_2)$ is a free R_2 module on even [odd] dimensional generators.

In light of this fact one defines a space pair (X, A) to be (2-even) [(2-odd)] if and only if $H_*(X, A; R_2)$ is a free R_2 module on even [odd] dimensional generators.

LEMMA 5.2. *Let* $G = Z_{2^k}$. *If* (X, A) *is a* G *pair and if* (X, A) *is* (2-*even*) [(2-*odd*)], *then* $\Omega_*^G(\mathcal{F}_e)(X, A) \otimes R_2$ *is a free* $\Omega_* \otimes R_2$ *module on even* [*odd*] *dimensional generators.*

PROOF. From Proposition 2.3 in [10] one learns that $\Omega_*^G(\mathcal{F}_e)(X, A) \cong \Omega_*(X \times_G EG, A \times_G EG)$ where EG is the total space of the universal principal G bundle. From the discussion preceding this lemma one learns that $(X \times_G EG, A \times_G EG)$ is (2-even) [(2-odd)]. As in [11, p. 145] one can show that if (X, A) is a c.w. pair such that $H_*(X, A; R_2)$ is a torsion free R_2 module, then

$$\Omega_*(X, A) \otimes R_2 \cong (\Omega_* \otimes R_2) \otimes_{R_2} H_*(X, A; R_2).$$

This yields the desired result. □

Thus it is of interest to examine the homology of the spaces introduced in Theorem 5.1. From the homology exact sequence of the cofibration $S(\det \gamma_{2s}) \to D(\det \gamma_{2s}) \to T(\det \gamma_{2s})$ in which BSO_{2s} is homotopy equivalent to $S(\det \gamma_{2s})$ and BO_{2s} is homotopy equivalent to $D(\det \gamma_{2s})$ one learns that $T(\det \gamma_{2s})$ is (2-odd). From the proof of Proposition 4.1 one knows that for $Z_{2^j} < Z_{2^{k-1}}$,

$$F_{Z_{2^j}}(T(\det \gamma_{2s})) = T(\det \gamma_{2s}/F_{Z_{2^j}}[F^-_{Z_2}(BO_{2s}, Z_2k)]).$$

If $E \to X$ is an oriented bundle, $\det E$ is a trivial line bundle and thus $T(\det E) = \Sigma X^+$ where Σ denotes reduced suspension. Now by Proposition 3.3, $F_{Z_{2^j}}[F^-_{Z_2}(BO_{2s}, Z_{2^k})]$ is homotopic to $\bigcup BU_{(t)}$ where (t) is a q-tuple of nonnegative integers $(t_1, t_2, \cdots, t_q)$ and $BU_{(t)} = BU_{t_1} \times BU_{t_2} \times \cdots \times BU_{t_q}$. Since $\gamma_{2s}/BU_{(t)}$ is complex, $\det \gamma_{2s}/BU_{(t)}$ is trivial and $T(\det \gamma_{2s}/\bigcup BU_{(t)}) = \bigvee \Sigma BU^+_{(t)}$. It follows that $F_{Z_{2^j}}(T(\det \gamma_{2s}))$ is (2-odd) for $0 \leqslant j \leqslant k-1$. Now from the appropriate cofibrations $X \vee Y \to X \times Y \to X \wedge Y$ one reads off the result:

LEMMA 5.3. *If* $F_{Z_{2^j}}(X, A)$ *is (2-even) [(2-odd)] for* $0 \leqslant j \leqslant k$ *and if* $Y = F_{Z_2}(X)/F_{Z_2}(A) \wedge T(\det \gamma_{2s})$, *then* Y *is a space with* $Z_{2^{k-1}}$ *action such that* $F_{Z_{2^j}}(Y)$ *is (2-odd) [(2-even)] for* $0 \leqslant j \leqslant k-1$. □

This brings one finally to the computations.

THEOREM 5.4. *If* $F_{Z_{2^j}}(X, A)$ *is (2-even) [(2-odd)] for* $0 \leqslant j \leqslant k$, *then* $\Omega_*^{Z_{2^k}}(\mathcal{F})(X, A) \otimes R_2$ *is a free* $\Omega_* \otimes R_2$ *module on even [odd] dimensional generators.*

PROOF. If $k = 0$ then both the even and the [odd] case follow from Lemma 5.2. Assume that the theorem is true for $k' < k$. Let (X, A) have a Z_{2^k} action satisfying the hypotheses. By Lemma 5.2, $\Omega_*^{Z_{2^k}}(\mathcal{F}_e)(X, A) \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even [odd] dimensional generators. Theorem 5.1 yields that

$$\Omega_*^{Z_{2^k}}(\mathcal{F}, \mathcal{F}_e)(X, A) \cong \bigoplus \widetilde{\Omega}_{*-2s+1}^{Z_{2^{k-1}}}(\mathcal{F})(F_{Z_2}(X)/F_{Z_2}(A) \wedge T(\det \gamma_{2s})).$$

By Lemma 5.3 and induction hypothesis, this implies that $\Omega_*^{Z_{2^k}}(\mathcal{F}, \mathcal{F}_e)(X, A) \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even [odd] dimensional generators.

Now consider the exact triangle

$$\Omega_*^{Z_{2^k}}(\mathcal{F}_e)(X, A) \otimes R_2 \to \Omega_*^{Z_{2^k}}(\mathcal{F})(X, A) \otimes R_2 \to \Omega_*^{Z_{2^k}}(\mathcal{F}, \mathcal{F}_e)(X, A) \otimes R_2.$$

∂'_* of degree -1

Note that it is in fact a split short exact sequence. This gives the induction step. □

Note. If $(X, A) = (pt, \varnothing)$, Theorem 5.4 says that $\Omega_*^{Z_{2^k}} \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even dimensional generators.

Note. This is the best possible result in the following sense. In [3, p. 105] P. E. Conner computes the torsion of $\Omega_*^{Z_2}$. There is too much torsion for $\Omega_*^{Z_2}$ to be a free Ω_* module.

Note. In the paper as originally submitted the author asserted that $\Omega_*^G \otimes R_2$ is a free $\Omega_* \otimes R_2$ module for G any finite cyclic group. However, the referee kindly noted a logical error in the author's proof of this statement. Nonetheless it is still a very reasonable conjecture, and in fact seems to be true in certain special cases (e.g. $Z_2 \times Z_p$). Along this line it should also be noted that in the author's dissertation [13] he proves via a somewhat arduous and noninstructive argument that for G a finite cyclic group the torison of Ω_*^G is all 2-torsion.

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